Online Supplement

This supplement proves the characterization in Appendix C.

Recall that consumers' idiosyncratic tastes (θ_i's) follow the distribution F shown in Figure 1S(a). Note that, in order for the distribution to be properly specified, the following two things must be true: (1) $a \leq \frac{1}{v_1+v_2}$ and (2) $b = \frac{1-av_2}{v_1}$.

We can calculate $F^{-1}(\cdot)$ for some critical points:

$$\begin{aligned}
&F^{-1}(0) = -\frac{b}{2} = -\frac{1-av_2}{2v_1},\\
&F^{-1}(\frac{b-a}{2}v_1) = F^{-1}(\frac{1}{2}(1-a(v_1+v_2))) = -\frac{a}{2},\\
&F^{-1}(\frac{b+a}{2}v_1 + av_2) = F^{-1}(\frac{1}{2}(1+a(v_1+v_2))) = \frac{a}{2},\\
&F^{-1}(1) = \frac{b}{2} = \frac{1-av_2}{2v_1}.
\end{aligned}$$

Equation (10) from the paper gives a formula for inverse demand:

$$\Delta = F^{-1}(1-Q_1) + \alpha(2Q_1 - 1) + \mu.$$

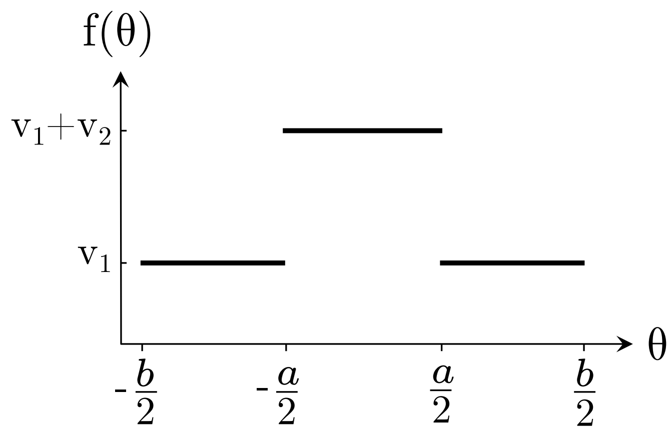
We can use equation (10) to calculate inverse demand at some critical points:

$$\begin{aligned}
&\text{At } Q_1 = 0 : \Delta = \frac{1-av_2}{2v_1} - \alpha + \mu.\\
&\text{At } Q_1 = \frac{1}{2}(1-a(v_1+v_2)) : \Delta = a[\frac{1}{2} - \alpha(v_1+v_2)] + \mu.\\
&\text{At } Q_1 = \frac{1}{2}(1+a(v_1+v_2)) : \Delta = a[-\frac{1}{2} + \alpha(v_1+v_2)] + \mu.\\
&\text{At } Q_1 = 1 : \Delta = \frac{-1+av_2}{2v_1} + \alpha + \mu.
\end{aligned}$$

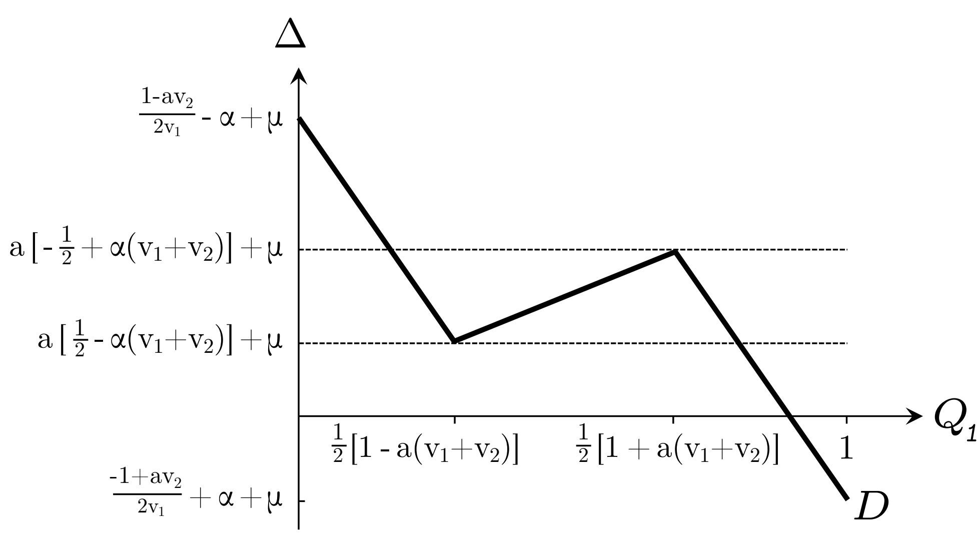
Figure 1S(b) plots the corresponding inverse demand curve.

Observe that, in order for demand to have an in/out shape, the following two conditions must be satisfied:

Figure 1S



(a) Pdf that gives rise to piecewise linear demand.



(b) Corresponding demand curve for the competitive case (demand is in/out if $\alpha_{\min} < \alpha < \alpha_{\max}$).

1. $a[\frac{1}{2} - \alpha(v_1 + v_2)] + \mu < a[-\frac{1}{2} + \alpha(v_1 + v_2)] + \mu$.

   $\iff \alpha > \frac{1}{2(v_1+v_2)} \equiv \alpha_{\min}$.

2. $\frac{1-av_2}{2v_1} - \alpha + \mu > a[-\frac{1}{2} + \alpha(v_1 + v_2)] + \mu$.

   $\iff \alpha < \frac{1+a(v_1-v_2)}{2v_1(1+a(v_1+v_2))} \equiv \alpha_{\max}$.

Therefore, we will focus attention on the case where $\alpha_{\min} < \alpha < \alpha_{\max}$.

Let $\tilde{\Delta} = \Delta - \mu$ denote the effective price differential. By assumption, firm 1 faces an "in" demand curve. It is easy to show that demand for good 1 can be written as follows:

$$Q_1(\tilde{\Delta}) = \begin{cases} z_{in} - s\tilde{\Delta}, & \text{if } \tilde{\Delta} \leq a(-\frac{1}{2} + \alpha(v_1 + v_2)) = \Delta_{\max}, \\ z_{out} - s\tilde{\Delta}, & \text{if } \tilde{\Delta} > \Delta_{\max}, \end{cases}$$

where $s = \frac{v_1}{1-2\alpha v_1}$, $z_{in} = \frac{1+av_2-2\alpha v_1}{2(1-2\alpha v_1)}$, and $z_{out} = \frac{1-av_2-2\alpha v_1}{2(1-2\alpha v_1)}$.

Notice that demand for firm 2 is:

$$Q_2(\tilde{\Delta}) = 1 - Q_1(\tilde{\Delta}).$$

With expressions in hand for demand, we are now in a position to analyze the pricing game by backward induction.

Stage 2: Firm 2 chooses p_2.

Let us determine firm 2's optimal choice of price given firm 1's price. We can define three prices:

1. $p_2^{kink} = p_1 - \mu - \Delta_{\max}^+$ denotes the value of p_2 such that $\tilde{\Delta} = \Delta_{\max}^+$.

2. p_2^- denotes the optimal price below p_2^{kink}.

3. p_2^+ denotes the optimal price above p_2^{kink}.

When firm 2 sets a price of p_2^{kink}, its profits are:

$$\pi_2^{kink} = (p_1 - \mu - \Delta_{\max})(1 - z_{out} + s\Delta_{max}).$$

Observe that p_2^- is the value of p_2 that maximizes $p_2 \cdot (1 - z_{out} + s\tilde{\Delta})$. Taking a first-order condition, we find that $p_2^- = \frac{p_1-\mu}{2} + \frac{1-z_{out}}{2s}$. The profits associated with price p_2^- are $\pi_2^- = p_2 \cdot (1 - z_{out} + s\tilde{\Delta})$.

Similarly, p_2^+ is the value of p_2 that maximizes $p_2 \cdot (1 - z_{in} + s\tilde{\Delta})$. Taking a first-order condition, we find that $p_2^+ = \frac{p_1-\mu}{2} + \frac{1-z_{in}}{2s}$. The profits associated with price p_2^+ are $\pi_2^+ = p_2 \cdot (1 - z_{in} + s\tilde{\Delta})$.

It can be shown that $\pi^- \geq \pi^{kink}$ if and only if:

$$p_1 \geq \mu + \frac{1 + a(v_2 - 2v_1)}{2v_1} + \alpha(2a(v_1 + v_2) - 1) \equiv \kappa_1.$$

It can also be shown that $\pi^{kink} \geq \pi^+$ if and only if:

$$\begin{aligned} p_1 \geq \mu + \frac{a(3v_2 - 2v_1) + 1}{2v_1} + \alpha(2a(v_1 + v_2) - 1) \\ - \frac{2}{v_1}\sqrt{av_2[\frac{1}{2}(1 + a(v_2 - v_1)) - \alpha v_1(1 - a(v_1 + v_2))]} \equiv \kappa_2. \end{aligned}$$

Two caveats should be noted. First, Q_2 cannot exceed 1. Consequently, if p_1 is sufficiently high ($p_1 > \mu + \frac{1+z_{out}}{s} \equiv \kappa_0$), the formula obtained from the first-order condition does not characterize p_2^-; instead $p_2^- = p_1 - \frac{z_{out}}{s} - \mu$.[1] Second, firm 2 will never set p_2 below zero. Consequently, if p_1 is sufficiently low ($p_1 \leq \mu + \frac{z_{in}-1}{s} \equiv \kappa_3$), the formula obtained from the first-order condition does not characterize p_2^+; instead, $p_2^+ = 0$.[2]

It is easy to show that $\kappa_0 > \kappa_1 > \kappa_2 > \kappa_3$. We conclude the following.

Region 1: If $p_1 > \kappa_0$,

$p_2 = p_2^- = p_1 - \frac{z_{out}}{s} - \mu$; $\tilde{\Delta} = \frac{z_{out}}{s}$; and $Q_1 = 0$.

Region 2: If $\kappa_0 \geq p_1 > \kappa_1$,

[1]In this case, p_2^- is the maximum price such that $Q_2 = 1$.

[2]When $p_1 \leq R_3$, there is no positive price for good 2 which results in non-zero demand.

$p_2 = p_2^- = \frac{p_1-\mu}{2} + \frac{1-z_{out}}{2s}$; $\tilde{\Delta} = \frac{p_1-\mu}{2} - \frac{1-z_{out}}{2s}$; and $Q_1 = \frac{1+z_{out}}{2} - s(\frac{p_1-\mu}{2})$.

Region 3: If $\kappa_1 \geq p_1 > \kappa_2$,

$p_2 = p_2^{kink}$; $\tilde{\Delta} = \Delta^+_{\max}$; and $Q_1 = z_{out} - s\Delta^+_{\max}$.

Region 4: If $\kappa_2 \geq p_1 > \kappa_3$,

$p_2 = p_2^+ = \frac{p_1-\mu}{2} + \frac{1-z_{in}}{2s}$; $\tilde{\Delta} = \frac{p_1-\mu}{2} - \frac{1-z_{in}}{2s}$; and $Q_1 = \frac{1+z_{in}}{2} - s(\frac{p_1-\mu}{2})$.

Region 5: If $\kappa_3 \geq p_1$,

$p_2 = 0$; $\tilde{\Delta} = \frac{z_{in}-1}{s}$; and $Q_1 = 1$.

Stage 1: Firm 1 chooses p_1.

To solve firm 1's problem, we will examine the profits from choosing a price in each of the five regions. It should be noted that firm 1's profits are continuous in p_1 *except* at $p_1 = \kappa_2$ (the boundary between regions 3 and 4) — since firm 2's price jumps discontinuously at $p_1 = \kappa_2$.

Region 1:

If firm 1 chooses a price in region 1 ($p_1 > \kappa_0$), $Q_1 = 0$. Hence, the firm earns zero profit:

$$\pi_1^{R1} = 0.$$

Region 2:

In region 2, firm 1's profits are: $p_1(\frac{1+z_{out}}{2} - s(\frac{p_1-\mu}{2}))$.

Provided the first-order condition holds, firm 1's optimal price is $p_1^{R2} = \frac{1+z_{out}}{2s} + \frac{\mu}{2} = \frac{3-av_2}{4v_1} - \frac{3}{2}\alpha + \frac{\mu}{2}$. The associated profits are:

$$\pi_1^{R2} = 2s(\frac{1+z_{out}}{2s} + \frac{\mu}{2})^2.$$

Region 3:

In region 3, firm 1's profits are $p_1(z_{out} - s\Delta^+_{\max})$. Hence, the firm optimizes by choosing the maximum possible price: $p_1^{R3} = \kappa_1$. The associated profits are:

$$\pi_1^{R3} = \kappa_1(z_{out} - s\Delta^+_{\max}).$$

Region 4:

In region 4, firm 1's profits are: $p_1(\frac{1+z_{in}}{2} - s(\frac{p_1-\mu}{2}))$. We can separate our analysis into two cases.

Case (i):

In case (i), firm 1's optimal price is determined by the first-order condition: $p_1^{R4i} = \frac{1+z_{in}}{2s} + \frac{\mu}{2} = \frac{3+av_2}{4v_1} - \frac{3}{2}\alpha + \frac{\mu}{2}$. The corresponding profits are:

$$\pi_1^{R4i} = 2s(\frac{1+z_{in}}{2s} + \frac{\mu}{2})^2.$$

In order for the optimal price to be determined by the first-order condition, it must be the case that: $p_1^{R4i} \leq \kappa_2$. This condition can be rewritten as:

$$\mu \geq \frac{1-a(5v_2-4v_1)}{2v_1} - \alpha(4a(v_1+v_2)+1) + \frac{4}{v_1}\sqrt{av_2[\frac{1}{2}(1+a(v_2-v_1)) - \alpha v_1(1-a(v_1+v_2))]}. \quad (*)$$

Case (ii):

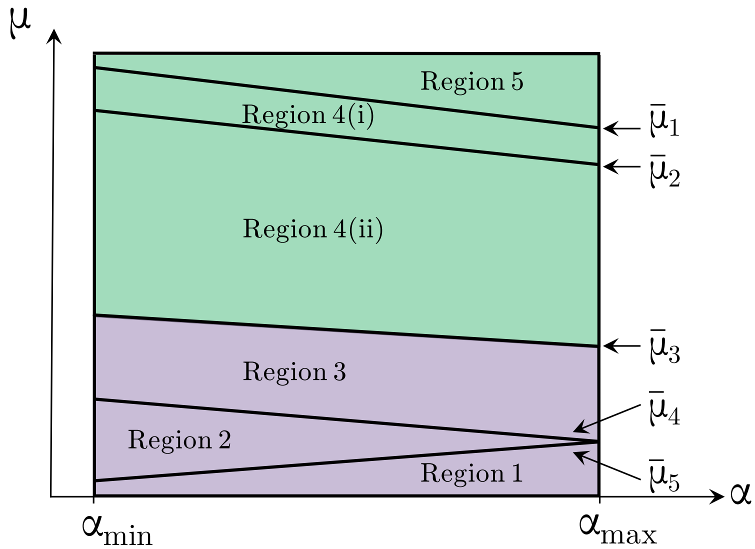
The optimal price is $p_1^{R4ii} = \kappa_2$. Firm 1's profits are equal to $\pi_1^{R4ii} = p_1^{R4ii} \cdot Q_1^{R4ii}$, where:

$$Q_1^{R4ii} = \frac{1-a(v_2-v_1)}{2(1-2v_1\alpha)} - \frac{\alpha v_1(a(v_1+v_2)+1)}{1-2v_1\alpha} + \frac{1}{1-2v_1\alpha}\sqrt{av_2[\frac{1}{2}(1+a(v_2-v_1)) - \alpha v_1(1-a(v_1+v_2))]}.$$

In case (ii), $\tilde{\Delta} = \Delta_{\max}$, which means that the remain-in constraint (RIC) is binding.

Region 5:

Figure 2S: The region where p_1^* lies.



In region 5, firm 1's profits are equal to p_1 (since $Q_1 = 1$). Hence, the firm optimizes by choosing the maximum possible price: $p_1^{R5} = \kappa_3 = \frac{-1+av_2+}{2v_1} + \alpha + \mu$. The associated profits are:

$$\pi_1^{R5} = \kappa_3.$$

The optimal price (p_1^*):

Figure 2S, drawn to scale for the case where $a = 1$ and $v_1 = v_2 = \frac{1}{3}$, shows the region where the optimal price, p_1^*, lies as a function of the parameters. Our remaining task is to derive the cutoff values of μ shown in the figure: $\overline{\mu}_1$, $\overline{\mu}_2$, $\overline{\mu}_3$, $\overline{\mu}_4$, and $\overline{\mu}_5$.

Cutoff $\overline{\mu}_1$:

Cutoff $\overline{\mu}_1$ is the value of μ for which $p_1^{R4i} = \kappa_3$. Solving this equation, we find that:

$$\overline{\mu}_1 = \frac{5 - av_2}{2v_1} - 5\alpha.$$

Cutoff $\overline{\mu}_2$:

Observe that $\overline{\mu}_2$ is the cutoff between cases (i) and (ii) in region 4. Equation (*)

gives a formula for $\overline{\mu}_2$:

$$\overline{\mu}_2 = \frac{1 - a(5v_2 - 4v_1)}{2v_1} - \alpha(4a(v_1 + v_2) + 1) + \frac{4}{v_1}\sqrt{av_2[\frac{1}{2}(1 + a(v_2 - v_1)) - \alpha v_1(1 - a(v_1 + v_2))]}.$$

Cutoff $\overline{\mu}_3$:

Cutoff $\overline{\mu}_3$ is the value of μ for which $\pi_1^{R4ii} = \pi_1^{R3}$.[3]

$$\overline{\mu}_3 = \frac{1 - a(5v_2 - 4v_1)}{2v_1} - \alpha(1 + 4a(v_1 + v_2)) + \frac{1 + 3a(v_2 - v_1) - 2\alpha v_1(1 - 3a(v_1 + v_2))}{v_1[1 + a(v_2 - v_1) + 2\alpha v_1(-1 + a(v_1 + v_2))]} \times \sqrt{av_2[\frac{1}{2}(1 + a(v_2 - v_1)) - \alpha v_1(1 - a(v_1 + v_2))]}.$$

Cutoff $\overline{\mu}_4$:

Cutoff $\overline{\mu}_4$ is the value of μ for which $p_1^{R2} = \kappa_1$. Solving this equation, we find that:

$$\overline{\mu}_4 = \frac{1 - a(3v_2 - 4v_1)}{2v_1} - \alpha(4a(v_1 + v_2) + 1).$$

Cutoff $\overline{\mu}_5$:

Cutoff $\overline{\mu}_5$ is the value of μ for which $p_1^{R2} = \kappa_0$. Solving this equation, we find that:

$$\overline{\mu}_5 = \frac{-3 + av_2}{2v_1} + 3\alpha.$$

This completes our analysis of the game. Figure 3S, again drawn to scale for the case where $a = 1$ and $v_1 = v_2 = \frac{1}{3}$, summarizes.

[3]Because firm 1's profits are discontinuous at $p_1 = \kappa_2$ (the boundary between regions 3 and 4), our technique to solve for $\overline{\mu}_3$ differs form our technique for other cutoffs.

Figure 3S: The equilibrium outcome.